MATH 380: Homework 2

1.

I. Prove the following combinatorial identity in two ways:

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}$$

(a) algebraically (using Pascal's triangle), and
(b) combinatorially (by counting things in different ways).

II. Prove the following identity combinatorially (by counting something):

$$\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}.$$

I. (a) Using Pascal's Identity, we can write:

$$\begin{aligned}&\binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}\\ =&\left(\binom{2n}{n+1} + \binom{2n}{n}\right) + \left(\binom{2n}{n} + \binom{2n}{n-1}\right)\\ =&\binom{2n+1}{n+1} + \binom{2n+1}{n}\end{aligned}$$

We can then apply Pascal's Identity one more time:

$$\begin{aligned}&\binom{2n+1}{n+1} + \binom{2n+1}{n}\\ =&\binom{2n+2}{n+1}\end{aligned}$$

This is precisely the formula we wanted. □

(b) Suppose we wish to count the number of unique strings of digits where each digit is either 0 or 1, there are $2n+2$ total digits, and $n+1$ of the digits are 1s. We do this in two different ways.

First, we count directly by choosing $n+1$ of the $2n+2$ place values to hold a 1. This results in a count of $\binom{2n+2}{n+1}$.

Next, we break the task into cases by fixing the first two digits in the string. If the first two digits are both 0, then we have $2n$ remaining digits to place, of which $n+1$ must be 1s – this can happen in $\binom{2n}{n+1}$ ways. If the first two digits are 01, then n of the remaining

$2n$ digits must be 1s – this can happen in $\binom{2n}{n}$ ways. Similarly, if the first two digits are 10, then n of the remaining $2n$ digits must be 1s, which again can happen in $\binom{2n}{n}$ ways. Finally, if the first two digits are both 1, then only $n-1$ of the remaining $2n$ digits must be 1s – this can happen in $\binom{2n}{n-1}$ ways. These are all the possible cases, and they are mutually exclusive, so we add up the choices for each to get a total of

$$\begin{aligned} &\binom{2n}{n+1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n-1} \\ =&\binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1} \end{aligned}$$

ways to arrange the $2n+2$ digits. We can set this equal to our previous expression to get

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}$$

which is what was wanted. □

II. Suppose we have 3 indistinguishable balls and a line of $n \geq 3$ boxes, and we wish to count the number of ways there are to choose 3 of the boxes to each hold one ball. We will do this in two different ways.

First, we count directly by saying that there are $\binom{n}{3}$ ways to choose 3 boxes out of the n.

Next, we break the task up as follows: Choose 1 of the n boxes to hold the "middle" ball, i.e. the one for which there will be a ball in a box somewhere to the left and another somewhere to the right; suppose we pick this "middle" box to be the ith from the left. Then there are $i-1$ boxes to the left of the middle box, and thus $\binom{i-1}{1} = i-1$ ways to pick the box for the leftmost ball; and there are $n-i$ boxes to the right of the middle box, and thus $\binom{n-i}{1} = n-i$ ways to pick the box for the rightmost ball. For any choice of i, then, there are $(i-1)(n-i)$ ways to place the remaining two balls. The cases are all distinct, since no choice of 3 boxes with middle box i_1 can be identical to a choice with middle box i_2; they also cover all the possibilities, since for any choice of 3 boxes, there must be a box in between the other two. Thus, we can sum up the possible choices for each of the cases from $i=1$ to $i=n$ to get the total number of ways to choose 3 boxes:

$$\sum_{i=1}^{n} (i-1)(n-i)$$

Setting this equal to our previous expression gives us the desired identity. □

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2. How many ways are there to represent a positive integer n as a sum of

(a) k non-negative integers?

(b) k positive integers?

Note: the order of summation matters. For example, take $n = 3$, $k = 2$. Then the possible sums in (a) are: $3 + 0, 2 + 1, 1 + 2, 0 + 3$.

(a) Write n as a sum of n 1s. Writing n as a sum of k non-negative integers is then a matter of placing $k - 1$ dividers among those n 1s, with placements not necessarily unique and placements on either end of the n 1s allowed: each set of 1s between consecutive dividers is summed to get one of the k non-negative integers. Placing $k - 1$ dividers among n objects is a balls-in-urns problem: there are $\binom{n+k-1}{n}$ ways to make that choice, and thus there are

$$\binom{n+k-1}{n}$$

ways to write n as a sum of k non-negative integers. □

(b) Placing the restriction that each of the k integers be positive effectively means that there must be at least one 1 between each of the dividers and no dividers can be on either end of the line of 1s. This then comes down to a problem of choosing $k - 1$ of the $n - 1$ spaces between 1s to put dividers in (i.e. placements of the dividers now cannot overlap and cannot be on either end of the n 1s): there are

$$\binom{n-1}{k-1}$$

ways in which this can be done. □

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3. Say that an integer is "complete" if its decimal representation contains all the digits $0, 1, \ldots, 9$. How many integers with up to n decimal digits are complete?

For $n < 10$, 0 integers are complete, since they are not long enough to contain all the digits $0, 1, \ldots, 9$. We thus focus our attention on integers with $n \geq 10$, and proceed by applying the Principle of Inclusion and Exclusion.

We start by noting that there are 10^n integers with up to n decimal digits. Clearly, not all of these are complete; we thus subtract all those that are not complete due to not having a 0 in their decimal representation (there are 9^n of them, since we have 9 remaining digit options for each of the n place values), those that are not complete due to not having a 1 in their decimal representation (there are 9^n of these also), those that are not complete due to not having a 2 (9^n of these, again), and so on, through those that do not have a 9 in their decimal representation. This brings us down to $10^n - 10 \cdot 9^n$.

However, we have double-counted the non-complete integers, since a number can be non-complete due to multiple digit deficiencies, so for each pair of digits in $0, 1, \ldots, 9$, we must add back the integers that are not complete due to lacking these digits; there are $\binom{10}{2}$ such digit pairs, and 8^n corresponding integers for each (since we have 8 remaining digit options for each of the n place values), so our expression becomes $10^n - 10 \cdot 9^n + \binom{10}{2} \cdot 8^n$.

Again, we have overcounted: integers lacking 3 digits in their decimal representation are subtracted three times when we subtract $10\cdot 9^n$ and then added back three times when we add $\binom{10}{2}\cdot 8^n$, but never subtracted off once more to get the desired result. Thus, for every set of three digits, we subtract from our tally the number of integers that are missing these three digits; there are $\binom{10}{3}$ such sets of three digits, and 7^n integers corresponding to each, so we can update our expression for the number of complete integers to $10^n - 10\cdot 9^n + \binom{10}{2}\cdot 8^n - \binom{10}{3}\cdot 7^n$.

Continuing in this manner, we conclude that the total count of complete integers with n or fewer digits in their decimal representation is

$$\begin{aligned} &10^n - \binom{10}{1}\cdot 9^n + \binom{10}{2}\cdot 8^n - \binom{10}{3}\cdot 7^n + \cdots - \binom{10}{9}\cdot 1^n \\ &= 10^n - \sum_{i=1}^{9}(-1)^i\binom{10}{i}\cdot(10-i)^n. \square \end{aligned}$$

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4. Let Γ be a simple connected graph (simple means Γ does not have loops or multiple edges). Show that any two longest paths on Γ must share a vertex.

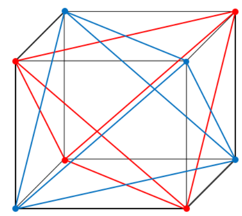
We prove this result by contradiction. Suppose $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sequences of vertices that define two longest paths in Γ, and suppose further that $a_p \neq b_q$ for all $1 \leq p, q \leq n$ (i.e. that paths a and b do not share a vertex). Since Γ is connected, there exists a shortest path from a_1 to b_1. This path must diverge from $a_1, a_2, \ldots, a_n$ at some a_i (with $1 \leq i \leq n$), and it must join $b_1, b_2, \ldots, b_n$ at some b_j (with $1 \leq j \leq n$). Since we know $a_i \neq b_j$ by our previous assumption that paths a and b do not share a vertex, we must connect a_i and b_j via some series of 0 or more vertices $c_1, c_2, \ldots, c_m$ that do not lie on either path (note that a_i and b_j could be connected simply by an edge with no other vertices in between – in this case, $m = 0$).

We now note that $a_1, \ldots, a_n$ contains $n-1$ edges, so one of $a_1, a_2, \ldots, a_i$ and $a_i, a_{i+1}, \ldots, a_n$ must contain at least $\frac{n-1}{2}$ edges; WLOG, let this be $a_1, a_2, \ldots, a_i$. Similarly, one of $b_1, b_2 \ldots, b_j$ and $b_j, b_{j+1}, \ldots, b_n$ must contain at least $\frac{n-1}{2}$ edges; WLOG, let this be $b_j, b_{j+1}, \ldots, b_n$. The path $a_i, c_1, \ldots, c_m, b_j$ contains $m+1$ edges. Putting this together, we see that there is a path $a_1, \ldots, a_i, c_1, \ldots, c_m, b_j, \ldots, b_n$ in Γ consisting of at least $\frac{n-1}{2} + (m+1) + \frac{n-1}{2} = n+m$ edges, which is longer than either $a_1, a_2, \ldots, a_n$ or $b_1, b_2, \ldots, b_n$. This contradicts our assumption that paths a and b are two longest paths in Γ, so we conclude that the two paths must share at least a vertex. $\square$

EC 1. Let X be a set of points in an n-dimensional plane ($n \geq 3$) such that all points in X have coordinates ± 1. Show that if the cardinality of X is bigger than $2^{n+1}/n$ then there exist three points in X which form an equilateral triangle.

The set of all possible points that could be in X defines an n-dimensional cube. We show the desired result by induction.

For the base case, consider $n = 3$. Connect each diagonal along the face of the cube; this results in two interlocking tetrahedra, as shown below.



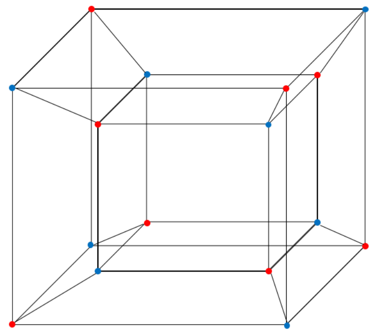
Note that we have divided the cube's vertices into two groups (red and blue) such that for each group, each vertex in the group is different from every other vertex in the group by at least two coordinates (which we can visualize as having a minimum distance of 2 edges from every other vertex in the group, if we travel only along the edges of the cube).

If we are given more than $\frac{2^{n+1}}{n} = \frac{16}{3} > 5$ points in this cube, then at least 3 of those points must lie in the same tetrahedron, by an extension of the Pigeonhole Principle. But any three points in a tetrahedron form an equilateral triangle, so the conclusion holds for our base case.

For our induction step, suppose that our conclusion holds for some integer $k \geq 3$. Then we have a k-dimensional cube for which choosing at least $\frac{2^{k+1}}{k}$ vertices will result in some three of them forming an equilateral triangle. Suppose furthermore that, as we did in the base case, we can group together vertices that are all mutually a minimum of 2 edges apart, thereby dividing the cube's vertices into two groups of equal size such that if any set of more than $\frac{2^k}{k}$ vertices is chosen from one of them, then among that chosen set there is guaranteed to be an equilateral triangle. We next form a $k+1$-dimensional cube by making two copies of the k-dimensional cube, giving each vertex in one of them a component of 1 in the $k+1$ dimension, and giving each vertex in the other a component of -1 in the $k+1$ dimension, then connecting the pairs of vertices that differ only in their $k+1$ component. (For a concrete visualization, imagine forming a hypercube from two cubes.)

Recall that for each k-dimensional cube, we categorized the vertices into two groups (red and blue). We now flip the color labels on one of the k-dimensional cubes from which the $k+1$-dimensional cube is built. This results in a grouping of the $k+1$-dimensional cube vertices that satisfies the same color properties as we previously defined them, since the distance from a red vertex in one k-dimensional cube to a red vertex in the other k-dimensional cube is a minimum of 2 edges (it must include at the minimum one edge traversal from one k-dimensional cube to the next, which includes a color flip, plus another edge traversal within one of the k-dimensional cubes, which includes a second color flip). Because of this color grouping, and the fact that each of these color groups forms a connected graph (which also follows by induction), we know that the vertices of the $k+1$-dimensional cube are comprised of those of two interlocking "hyper-tetrahedra" formed by connecting two "hyper-tetrahedra" one dimension lower. (Again, consider a hypercube as a concrete example – see below.)

Now choose some set X containing more than $\frac{2^{k+2}}{k+1}$ vertices in this $k+1$-dimensional cube. We know that one of the vertex groupings must have at least half this number of vertices, i.e. at least $\frac{2^{k+1}}{k+1}$ vertices. From here we break our $k+1$-dimensional "hyper-tetrahedron" into its two k-dimensional "hyper-tetrahedron" subcomponents (each one corresponding to one of the k-dimensional cubes), noting that one of these subcomponents must contain at least $\frac{2^k}{k+1}$ vertices, or



more than $\frac{2^k}{k}$ vertices. But by our induction hypothesis, this many vertices in such a k-dimensional "hyper-tetrahedron" corresponding to a k-dimensional cube guarantees the presence of an equilateral triangle.

We conclude that if we choose more than $2^{n+1}/n$ points from an n-dimensional space as described by the problem, there must exist three points in that set that form an equilateral triangle. $\square$